MATH 347 HW 11

due December 11, at the beginning of class

Homework Guildlines

Obviously, your solutions need to be complete and correct, but to receive full credit your write-up should also satisfy the following:

- All the important logical steps in the proof should be present and fully explained.
- All assumptions should be clearly identified.
- Your solutions should be clear and concise. If a sentence does not further the reader's understanding of the solution then it has no place in your write up.
- Use full and grammatically correct English sentences. Mathematical symbols should be used only to render complex mathematical relationships into a readable form.

Moreover, in order to obtain full credit for the homework, you must write down, in the very least, an attempt at a solution for each problem.

Problems

Do the following problems from your book: 7.6, 7.9, 7.17, 7.28, 7.34, 7.35. In addition to these problems please do the following:

(1) Consider the relation on $\mathbb{N} \times \mathbb{N}$ defined by

$$(a, b) \sim (c, d) :\iff a + d = b + c.$$

Show that

(a) $\sim$ defines an equivalence relation.

(b) Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}; (a, b) \mapsto a - b$. Show this respects the equivalence relation, that is

$$(a, b) \sim (c, d) \implies f(a, b) = f(c, d).$$

Conclude from this that there is a unique function $\varphi : \mathbb{N} \times \mathbb{N}/_\sim \to \mathbb{Z}$[1].

(c) Prove that φ defines a bijection.

(d) Note that $\mathbb{N} \times \mathbb{N}$ can be given an addition operation by

$$(a, b) + (c, d) = (a + c, b + d).$$

Show that this induces a well defined operation on the quotient set. Do the same for multiplication.

(e) Argue that φ satisfies the following condition [2]

$$\varphi([a, b] + [c, d]) = \varphi([a, b]) + \varphi([c, d]).$$

(2) Let $\mathcal{R}$ denote the set of Cauchy sequences in $\mathbb{Q}$. Let $\sim$ denote the relation on $\mathcal{R}$ given by

$$\{a_n\} \sim \{b_n\} :\iff \lim_{n\to\infty} a_n - b_n = 0$$

Please show the following:

(a) Show that $\sim$ is an equivalence relation on $\mathcal{R}$[3].

(b) Consider the function

$$f : \mathcal{R} \to \mathbb{R}; \{a_n\} \mapsto \lim_{n\to\infty} a_n$$

Show that this function respects the equivalence relation. Conclude that there is a unique map $\varphi : \mathcal{R}/\sim \to \mathbb{R}$.

(c) Show that φ is a bijection.

(d) One can define addition and multiplication on $\mathcal{R}$ by

$$\{a_n\} + \{b_n\} := \{a_n + b_n\} \qquad \{a_n\} \cdot \{b_n\} := \{a_n \cdot b_n\}.$$

Show that these induce well-defined operations on $\mathcal{R}/\sim$.

(e) Show that the map φ will satisfy

$$\varphi([\{a_n\}] + [\{b_n\}]) = \varphi([\{a_n\}]) + \varphi([\{b_n\}]).$$

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[1]This procedure is an instance of a more general construction referred to *group completion* or *Grothendieck completion*. It is a process which takes a *monoid* and produces a *group*.

[2]Thus φ is an example of a *group homomorphism*.

[3]The quotient set $\mathcal{R}/\sim$ is an instance of a procedure known as the *Cauchy completion* of $\mathbb{Q}$. It it can be applied more generally to any *metric space* (X, d).